Critical properties of the O(N) invariant scalar model using the auxiliary-mass method at finite temperature

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Using the auxiliary-mass method, the O(N) invariant scalar model is investigated at finite temperature. This mass and an evolution equation allow us to calculate an effective potential without an infrared divergence. A second order phase transition is indicated by the effective potential. The critical exponents are determined numerically. [S0556-2821(98)09418-1]

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Symmetry restoration of the O(N) scalar model at high temperature is very important since many physical systems belong to the same universality class: the polymer phase transition ($N\to 0$), critical liquid-vapor phase transition ($N=1$), alloy (e.g. β brass) phase transition ($N=1$), uniaxial ferromagnet phase transition ($N=1$), superfluid phase transition ($N=2$), ferromagnet phase transition ($N=3$), and chiral phase transition with two flavor massless quarks ($N=4$) [1,2].

The phase transition should be investigated by finite-temperature field theory, which is based on the statistical principle only. Perturbation theory, however, breaks down around the critical temperature when the phase transition is second or weakly first order [3], even if daisy diagrams (ring-diagrams) [4–6] are resumed. Investigation into phase transitions at finite-temperature has long been hampered by this failure. Many methods to avoid the failure were proposed: Cornwall-Jackiw-Tomboulis (CJT) method [7], renormalization improvement [8], novel summation [9], Padé improvement [10], exact renormalization group with temperature [11], and auxiliary-mass method [12–14].

The auxiliary-mass method is used in the present paper. The method is based on the following idea. First an effective potential can be calculated at large mass by the perturbation theory since it is reliable there. Second the effective potential is extrapolated to the small mass range, where the perturbation theory is not reliable, using an evolution equation. Finally various quantities (e.g. critical exponents) are determined from the effective potential. The method is applied to O(N) scalar model in three spatial dimensions concretely in the following; the phase transition of the model is investigated and the critical exponents are determined.

We consider the following Lagrangian density:

$$\mathcal{L}_E=-\frac{1}{2}\left(\frac{\partial\phi_a}{\partial\tau}\right)^2-\frac{1}{2}(\boldsymbol{\nabla}\phi_a)^2-\frac{1}{2}m^2\phi_a^2 -\frac{\lambda}{4!}(\phi_a^2)^2+J_a\phi_a+c.t., \quad (1)$$

where the subscript E refers to Euclidean, the index "a" runs from 1 to N, J_a is an external source function and $c.t.$ is the abbreviation of $counterterms$. First the effective potential V is calculated by the perturbation theory within one-loop order at large mass $m^2=M^2=O(T^2)$. We choose field expectation values $\overline{\phi_a}=\overline{\phi}\delta_{1a}$ without a loss of generality because of O(N) invariance:

$$\begin{aligned}V=&\frac{1}{2}M^2\overline{\phi}^2+\frac{\lambda}{4!}\overline{\phi}^4+\frac{T}{2\pi^2}\int_0^\infty drr^2\\&\times\log\left[1-\exp\left(-\frac{1}{T}\sqrt{r^2+M^2+\frac{\lambda}{2}\overline{\phi}^2}\right)\right]\\&+(N-1)\frac{T}{2\pi^2}\int_0^\infty drr^2\\&\times\log\left[1-\exp\left(-\frac{1}{T}\sqrt{r^2+M^2+\frac{\lambda}{6}\overline{\phi}^2}\right)\right]. \end{aligned}\quad (2)$$

We note that the daisy-resummation is not necessary because of the large mass and one-loop zero-temperature effect is negligible if the coupling is weak. Next this effective potential is extrapolated to smaller mass using the following evolution equation [13]:

$$\begin{aligned}\frac{\partial V}{\partial m^2}=&\frac{1}{2}\overline{\phi}^2+\frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{+i\infty+\epsilon}dp_0\\&\times\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{-p_0^2+\boldsymbol{p}^2+m^2+\frac{\lambda}{2}\overline{\phi}^2+\Pi_H}\\&\times\frac{1}{e^{p_0/T}-1}+\frac{N-1}{2\pi i}\int_{-i\infty+\epsilon}^{+i\infty+\epsilon}dp_0\\&\times\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{-p_0^2+\boldsymbol{p}^2+m^2+\frac{\lambda}{6}\overline{\phi}^2+\Pi_{NG}}\\&\times\frac{1}{e^{p_0/T}-1}.\end{aligned}\quad (3)$$

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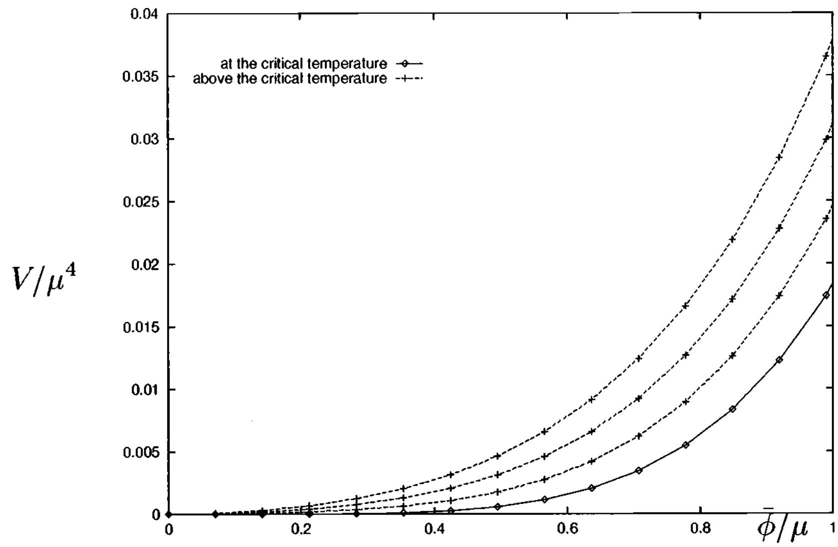

FIG. 1. The effective potential obtained by the auxiliary-mass method ($N=4$, $\lambda=1$). The dashed curves represent the effective potentials above the critical temperature and the solid line shows it at the critical temperature. Second-order phase transition occurs at the critical temperature. Similar behavior are observed for the other N and λ.

Here $\Pi_H=\Pi_H(m^2,\overline{\phi}^2,p_0^2,\mathbf{p}^2,T)$ and $\Pi_{NG}=\Pi_{NG}(m^2,\overline{\phi}^2,p_0^2,\mathbf{p}^2,T)$ are the full self-energies for ''Higgs mode'' and ''Nambu-Goldstone mode,'' that is, massive and massless modes in the broken phase respectively. This equation is modified from that of $\lambda\phi^4$ theory [13] straightforwardly through a diagonalization of the propagator. Though this equation is exact, it cannot be solved without an approximation; because it includes the full propagator which is not known exactly. We then replace it as follows:

$$m^2+\frac{\lambda}{2}\overline{\phi}^2+\Pi_H(0,0,\overline{\phi},m^2,\tau)\rightarrow\frac{\partial^2 V}{\partial\overline{\phi}^2}$$

$$m^2+\frac{\lambda}{6}\overline{\phi}^2+\Pi_{NG}(0,0,\overline{\phi},m^2,\tau)\rightarrow\frac{1}{\overline{\phi}}\frac{\partial V}{\partial\overline{\phi}}. \tag{4}$$

This replacement corresponds to setting an external momentum of the full self-energy to zero.[1] It allows us to convert Eq. (3) to the following partial differential equation:

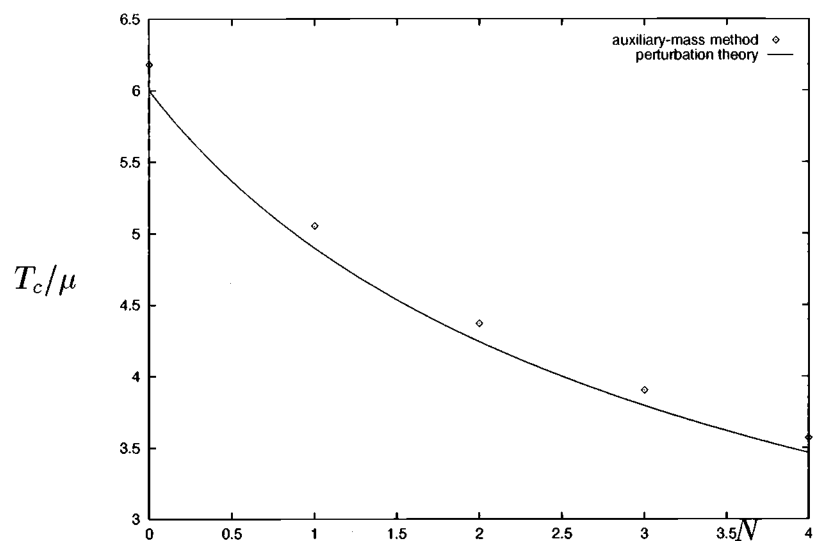

FIG. 2. Critical temperature as a function of N at $\lambda=1$ ($\diamond$). This resembles a result of the perturbation theory at leading order (—) but has a slight difference quantitatively.

[1]This is the first approximation of a systematic calculation [15].

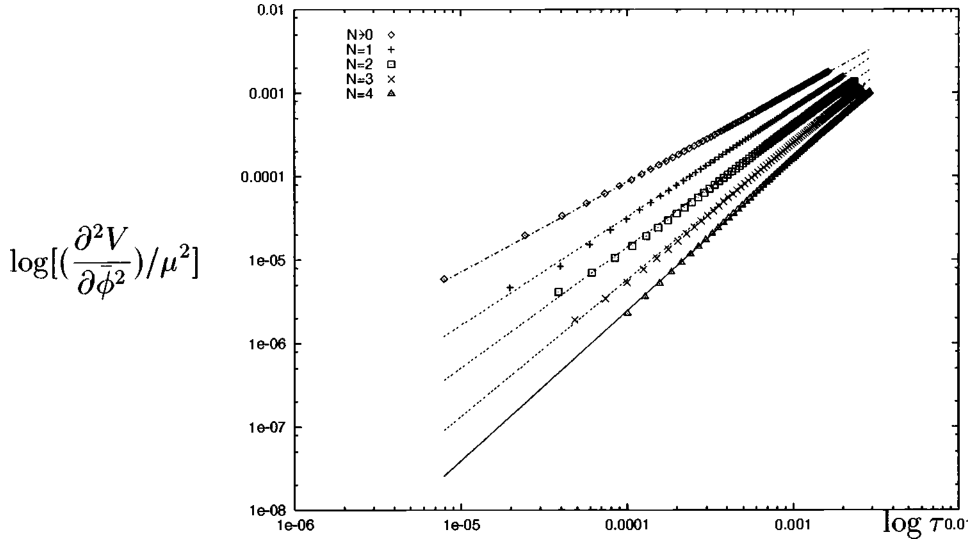

FIG. 3. Second derivative of the effective potential with respect to ϕ. The gradients are steeper for larger N.

$$\frac{\partial V}{\partial m^2}=\frac{1}{2}\,\overline{\phi}^2+\frac{1}{4\pi^2}\int_0^\infty dr r^2\,\frac{1}{\sqrt{r^2+\dfrac{\partial^2 V}{\partial\overline{\phi}^2}}}\times\frac{1}{\exp\left(\dfrac{1}{T}\sqrt{r^2+\dfrac{\partial^2 V}{\partial\overline{\phi}^2}}\right)-1}+\frac{N-1}{4\pi^2}\int_0^\infty dr r^2\frac{1}{\sqrt{r^2+\dfrac{1}{\overline{\phi}}\dfrac{\partial V}{\partial\overline{\phi}}}}\times\frac{1}{\exp\left(\dfrac{1}{T}\sqrt{r^2+\dfrac{1}{\overline{\phi}}\dfrac{\partial V}{\partial\overline{\phi}}}\right)-1}. \tag{5}$$

The effective potential V is numerically evolved from $m^2=M^2$ down to $m^2=-\mu^2$, where O(N) symmetry is broken at zero-temperature, under the initial condition Eq. (2). We solve it above the critical temperature only and can get sufficient information about the critical phenomenon.

The effective potentials are calculated using numerical methods in Ref. [14]. They are shown in Fig. 1. We cannot observe a second minimum which is a hallmark of a first order phase transition. Instead, the curvature of the origin decreases and vanishes smoothly as temperature decreases; one can, therefore, observe that the phase transition of the model is second order as it should be.

The critical temperature is shown as a function of N in Fig. 2. Though they resemble to a result of the perturbation theory at leading order [16], $T_c=6\sqrt{2/\lambda(N+2)}$, which is determined from the condition that mass with the daisy diagram vanishes, they have a slight difference quantitatively.

Finally the critical exponents, γ and δ, are determined using this potential. We determine γ from a second derivative of the effective potential with respect to $\overline{\phi}$ at the minimum. The critical exponent γ is defined as follows:

$$\chi\equiv\frac{\partial\overline{\phi}}{\partial J_1}\bigg|_{J_a=0}\sim\tau^{-\gamma}\quad[\tau=(T-T_c)/T_c]. \tag{6}$$

The following identity then relates the second derivative to the susceptibility:

$$\frac{\partial\overline{\phi}}{\partial J_1}\bigg|_{J_a=0}=\left(\frac{\partial^2 V}{\partial\overline{\phi}^2}\right)^{-1}\bigg|_{\overline{\phi}=\phi_c}. \tag{7}$$

Here ϕ_c is determined from the condition, $\partial V/\partial\overline{\phi}|_{\overline{\phi}=\phi_c}=0$. The second derivative are shown in log-scale in Fig. 3. We then determine the gradients, which is the very γ we want. One can observe that they become steeper as N increases; then, γ become larger. The results are $\gamma=1.13$ ($N\to 0$), 1.37 ($N=1$), 1.47 ($N=2$), 1.60 ($N=3$), 1.66 ($N=4$). These results are summarized in Table I, compared with the Landau approximation and world best values.

TABLE I. The critical exponents, γ and δ, obtained in the present paper. Those of Landau approximation (LA) and world best values (WBV) are also summarized. We used lattice results as WBV.

	γ (LA,WBV)	δ (LA,WBV)
$N\to 0$ [1]	1.13 (1, 1.16)	3.8 (3, 4.77)
$N=1$ [1]	1.37 (1, 1.24)	4.0 (3, 4.76)
$N=2$ [1]	1.47 (1, 1.32)	4.2 (3, 4.78)
$N=3$ [1]	1.60 (1, 1.40)	4.4 (3, 4.78)
$N=4$ [18]	1.66 (1, 1.48)	4.4 (3, 4.85)

We determine δ using the effective potential at the critical temperature. The critical exponent δ is defined as follows:

$$\overline{\phi} \propto J_1^{1/\delta} = \left(\frac{\partial V}{\partial \overline{\phi}} \right)^{1/\delta} \quad (T = T_c). \tag{8}$$

The following relation, which derived from Eq. (8), enables us to determine δ from the effective potential at the critical temperature:

$$V \propto \overline{\phi}^{\delta+1} \quad (T = T_c). \tag{9}$$

The effective potential at the critical temperature is shown in Fig. 1. We determine δ from it. The results are $\delta = 3.8$ ($N \to 0$), 4.0 ($N=1$), 4.2 ($N=2$), 4.4 ($N=3$), 4.4 ($N=4$).

In the present paper, we showed that the phase transition of O(N) scalar model is second order using auxiliary-mass method at finite temperature. This is great progress because we cannot show it using the perturbation theory with daisy resummation—the traditional method to calculate the effective potential at finite temperature [3].

Since the phase transition turned out to be second order, we determined the critical exponents of the phase transition. Though the results are not as accurate as world best values, they are much better than that obtained in the Landau approximation. The error would be due to the replacement (4). We must improve the approximation in order to get more accurate values [15].

In conclusion, the auxiliary-mass method turns out to be reliable not only qualitatively but also in rough quantitative estimation in O(N) invariant scalar model. This method would be reliable in the other models. What is more, we can investigate not only second-order phase transition but also first-order phase transition without any modifications [17]. This method, therefore, enables us to investigate the phase transition of various models: the cubic anisotropy model, the Abelian Higgs model, and the standard model.

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